Black-body laws derived from a minimum knowledge of Physics

A.G.Agnese[1], M.La Camera[2] and E.Recami[3]

[1,2]*Dipartimento di Fisica dell'Università di Genova*
Istituto Nazionale di Fisica Nucleare,Sezione di Genova
Via Dodecaneso 33, 16146 Genova, Italy
[3]*Facoltà di Ingegneria dell'Università di Bergamo*
Via Marconi 5, 24044 Dalmine (BG), Italy
Istituto Nazionale di Fisica Nucleare,Sezione di Milano

Abstract

Starting from the knowledge of the four fundamental quantities length L, mass M, time T, absolute temperature θ and accepting the validity of Gauss's law in all dimensions, we generalize, by the theory of physical dimensions, the expressions of the Stephan-Boltzmann law and of the Planck's formula for the black-body radiation to a space-time with one time and n space coordinates. In the particular case $n = 3$ we shall recover the known results.

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[1]E-mail: agnese@ge.infn.it
[2]E-mail: lacamera@ge.infn.it
[3]E-mail: erasmo.recami@mi.infn.it

1   Introduction

Applications of dimensional analysis to mechanical problems are well known in physics.

From a historical point of view, the first seeds of that method can be found in Newton's *"Principia"* [1] and in Fourier's *"Théorie Analytique de la Chaleur"* [2]. Successively, important contributions to the modern treatment of dimensional theory have been made by R.C.Tolman who defined the principle of dimensional homogeneity and stated the Π theorem [3].

It is worth noticing that dimensional analysis was employed by Albert Einstein too in the calculus of the eigenfrequencies in solid bodies [4]. He set forth the pendulum example and, on obtaining $T = C\sqrt{\dfrac{l}{g}}$ as usual, said: *"One can, as is known, get a little more out from dimensional considerations, but not with complete rigor. The dimensionless numerical factor (like C here), whose magnitude is only given by a more or less detailed mathematical theory, are usually of the order of unity. We cannot require this rigorously, for why shouldn't a numerical factor like $(12\pi)^3$ appear in a mathemathical-physical deduction? But without doubt such cases are rarities ... "* [5].

For a more complete view of the subject, we refer the reader to the book of Bridgman [6].

Our aim here is deducing the form of the Stephan-Boltzman law and

of the Planck's formula in a space-time with one time and n space coordinates, by employing the methods of dimensional analysis only. The minimum knowledge of physics we need getting started is given by the definitions of length L, mass M, time T and absolute temperature θ; we also accept the validity of Gauss's law in all dimensions and know the definitions of force and energy. Finally, in the particular case $n = 3$ we shall recover the known results on the basis of little physical information.

2 The universal constants and the Π theorem

Let us recall here some elements of physical dimensions theory. For instance, if we aim at a physical theory of gravity, the three fundamental units L, M and T are sufficient, but they must be associated as usual to some universal constants. In the gravitational case two universal constants are needed: an invariant velocity c with dimensions $[c] = [LT^{-1}]$ and a gravitation constant G_n, where the subscript n stands for the number of space coordinates, with physical dimensions $[G_n] = [L^n M^{-1} T^{-2}]$. The latter dimensional equality follows immediately from Gauss's law for the gravitational field $\boldsymbol{g}_n$ produced by a n-dimensional distribution of mass ρ_n:

$$\boldsymbol{\nabla} \cdot \boldsymbol{g}_n = -\frac{2\pi^{n/2}}{\Gamma(n/2)} G_n \rho_n \tag{1}$$

and from the relation

$$\boldsymbol{F}_n = m\boldsymbol{g}_n \tag{2}$$

which gives the force $\boldsymbol{F}_n$ (with dimensions $[LMT^{-2}]$) acting on a test mass m located in the gravitational field $\boldsymbol{g}_n$.

If the source is a point particle of mass M we have, in a n-dimensional space:

$$|\boldsymbol{g}_n| = \begin{cases} \dfrac{G_n M}{r^{n-1}} & \text{for} \quad n > 1 \\[2ex] \dfrac{G_1 M}{2} & \text{for} \quad n = 1 \end{cases} \tag{3}$$

In dealing with electromagnetism, we need again c and another fundamental constant.We do not introduce any additional physical quantities besides L, M and T; so, Gauss's law for the electrostatic field $\boldsymbol{E}_n$ produced by a n-dimensional distribution of charge ρ_n will be written, in Gaussian units, as:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E}_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \rho_n \tag{4}$$

Acting as in the gravitational case, the electrostatic field $\boldsymbol{E}_n$ of a point charge Q_n (whose dimensions turn out to be $[L^{n/2}M^{1/2}T^{-1}]$) is :

$$|\boldsymbol{E}_n| = \begin{cases} \dfrac{Q_n}{r^{n-1}} & \text{for} \quad n > 1 \\[2ex] \dfrac{Q_1}{2} & \text{for} \quad n = 1 \end{cases} \tag{5}$$

Before going on, we notice that in the three dimensional space Coulomb's law for two point charges $Z_1 e$ and $Z_2 e$

$$|\boldsymbol{F}| = \frac{Z_1 Z_2 e^2}{r^2} \qquad (6)$$

can be written, by introducing into it the fine structure constant $\alpha = e^2/(\hbar c)$, as

$$|\boldsymbol{F}| = \frac{Z_1 Z_2 \alpha \hbar c}{r^2} \qquad (7)$$

So the other fundamental constant we need can be either the electron charge e or the Planck's constant $\hbar$. Therefore when dealing within both gravity and electromagnetism, we can associate (L, M, T) with (c, G, e) or with $(c, G, \hbar)$: this agrees with the choices made, respectively by Stoney in 1874 [7] and by Planck in 1899 [8]. In our approach the appearance of the Planck's constant $\hbar$ is clearly related to the quantization of the electric charge, however.

Let us recall that, according to Planck, [8] the system of "natural units" must include $\sqrt{(\hbar c)/G}$ as unit of mass, $\sqrt{(\hbar G)/c^3}$ as unit of length and $\sqrt{(\hbar G)/c^5}$ as unit of time; while according to Stoney [7] the "natural units" for mass, length and time are $e/\sqrt{G}$, $(\sqrt{G}e)/c^2$ and $(\sqrt{G}e)/c^3$ respectively. One sees immediately that Stoney's units are given by Planck's times the square root of the fine structure constant. By the way, one may notice that Stoney's conception of the electron is somewhat different from that resulting from the subsequent discovery by Thomson.

Going back to our task, it is suitable, in n spatial dimensions, to pass from G to G_n and from α to $\alpha_n = e_n^2/(\hbar_n c)$; so, being α_n dimensionless by definition, the replacement of e with e_n and of $\hbar$ with $\hbar_n$ makes their respective dimensions to be $[e_n] = [L^{n/2}M^{1/2}T^{-1}]$ and $[\hbar_n] = [L^{n-1}MT^{-1}]$. Of course, the quantities labelled with n are defined only formally and not operationally, but this procedure will provide interesting results.

Let us fix our attention at this point on ordinary thermodynamics. In this case we need a fourth quantity, namely, the absolute temperature θ which then must be associated with another universal constant. In a quite natural way, we choose the Boltzmann constant k whose dimensions are $[L^2MT^{-2}\theta^{-1}]$.

Consequently in the dimensional equations which refer to the thermodynamics of the electromagnetic radiation factors of the type $c^\alpha \hbar_n^\beta k^\gamma$ must appear.

Let us finally recall the so called Π theorem [6] which we state as follows: "Denote by $P_1, P_2, \ldots P_r$ the magnitudes of quantities which may be physical magnitudes or experimental constants. Suppose that a functional relation $f(P_1, P_2, \ldots P_r) = 0$ holds, and furthermore that this is a complete equation and is the only relation between the magnitudes. Then, if there are s fundamental magnitudes the relation can be expressed in the form $F(\Pi_1, \Pi_2, \ldots \Pi_{r-s}) = 0$ where the Π's are the $r - s$ independent products

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of the arguments $P_1, P_2, \ldots P_r$ which are dimensionless in the fundamental magnitudes".

3   The Stefan-Boltzmann law

Now we are ready to establish the Stefan-Boltzmann law in a n-dimensional space. Denoting by $u_n(\theta)$ the energy density of the electromagnetic radiation enclosed in a cavity and in equilibrium at the temperature θ, the functional relation quoted in the Π theorem is $f(u_n, \theta, c, \hbar_n, k) = 0$. In this case $r - s = 5 - 4 = 1$, so there is only one product Π which can be written in the form:

$$\Pi_1 = u_n^{-1} c^{\alpha_1} \hbar_n^{\beta_1} k^{\gamma_1} \theta^{\delta_1} \tag{8}$$

Remembering that $[u_n] = [L^{2-n} M T^{-2}]$ and that consequently we shall have to discard the values $n = 1$ and $n = 2$ by physical reasons, we obtain the algebraic system:

$$\begin{cases} \beta_1 + \gamma_1 - 1 = 0 \\ \alpha_1 + (n-1)\beta_1 + 2\gamma_1 + n - 2 = 0 \\ \alpha_1 + \beta_1 + 2\gamma_1 - 2 = 0 \\ \gamma_1 - \delta_1 = 0 \end{cases} \tag{9}$$

For $n \geq 3$ the solutions are:

$$
\begin{cases}
\alpha_1 = -\dfrac{n}{n-2} \\[2mm]
\beta_1 = -\dfrac{n}{n-2} \\[2mm]
\gamma_1 = \dfrac{2(n-1)}{n-2} \\[2mm]
\delta_1 = \dfrac{2(n-1)}{n-2}
\end{cases}
\tag{10}
$$

and therefore

$$
u_n \propto \frac{(k\theta)^{2(n-1)/(n-2)}}{(\hbar_n c)^{n/(n-2)}}
\tag{11}
$$

When $n = 3$ we recover the Stephan-Boltzmann law:

$$
u \propto \frac{(k\theta)^4}{(\hbar c)^3}
\tag{12}
$$

Obviously the numerical factor $\pi^2/15$ which appears in that law cannot be determined by dimensional analysis.

4   The Planck's formula

Let us now obtain the Planck's formula in a n-dimensional space.

Calling $u_{n,\omega}(\omega, \theta)$ the spectral energy density of the black-body radiation in the frequency interval between ω and $\omega + d\omega$, the functional relation is now $f(u_{n,\omega}, \omega, \theta, c, \hbar, k) = 0$. In this case $r - s = 6 - 4 = 2$, so there are two products Π's which will be written as

$$
\Pi_2 = u_{n,\omega}^{-1} c^{\alpha_2} \hbar^{\beta_2} k^{\gamma_2} \omega^{\delta_2}
\tag{13}
$$

$$
\Pi_3 = \theta^{-1} c^{\alpha_3} \hbar^{\beta_3} k^{\gamma_3} \omega^{\delta_3}
\tag{14}
$$

and satisfy a relation of the form $F(\Pi_2, \Pi_3) = 0$.

The two algebraic systems one obtains are:

$$\begin{cases} \beta_2 + \gamma_2 - 1 = 0 \\[2mm] \alpha_2 + (n-1)\beta_2 + 2\gamma_2 + n - 2 = 0 \\[2mm] \alpha_2 + \beta_2 + 2\gamma_2 + \delta_2 - 1 = 0 \\[2mm] \gamma_2 = 0 \end{cases} \qquad \begin{cases} \beta_3 + \gamma_3 = 0 \\[2mm] \alpha_3 + (n-1)\beta_3 + 2\gamma_3 = 0 \\[2mm] \alpha_3 + \beta_3 + 2\gamma_3 + \delta_3 = 0 \\[2mm] \gamma_3 + 1 = 0 \end{cases} \tag{15}$$

with the respective solutions:

$$\begin{cases} \alpha_2 = -(2n-3) \\[2mm] \beta_2 = 1 \\[2mm] \gamma_2 = 0 \\[2mm] \delta_2 = 2n-3 \end{cases} \qquad \begin{cases} \alpha_3 = -(n-3) \\[2mm] \beta_3 = 1 \\[2mm] \gamma_3 = -1 \\[2mm] \delta_3 = n-2 \end{cases} \tag{16}$$

Therefore from the relation $F(\Pi_2, \Pi_3) = 0$ it follows that

$$u_{n,\omega} = \frac{\hbar_n \omega^{2n-3}}{c^{2n-3}} \, \varphi \left(\frac{\hbar_n \omega^{n-2}}{c^{n-3} k\theta} \right) \tag{17}$$

On the other hand it results, by definition, that $\int_0^\infty u_{n,\omega}(\omega, \theta)\, d\omega = u_n(\theta)$; so, by using the expressions already found for $u_{n,\omega}(\omega, \theta)$ and for $u_n(\theta)$, we obtain the following equation to be satisfied by the unknown function φ:

$$\int_0^\infty \frac{\hbar_n \omega^{2n-3}}{c^{2n-3}} \, \varphi \left(\frac{\hbar_n \omega^{n-2}}{c^{n-3} k\theta} \right) d\omega \propto \frac{(k\theta)^{2(n-1)/(n-2)}}{(\hbar_n c)^{n/(n-2)}} \tag{18}$$

Passing to the new variable $\varepsilon_n = (\hbar_n \omega^{n-2})/c^{n-3}$, whose meaning as a photon energy is apparent, the last equation becomes:

$$\frac{1}{(n-2)} \int_0^\infty \varepsilon_n^{2(n-1)/(n-2)-1} \varphi \left(\frac{\varepsilon_n}{k\theta} \right) d\varepsilon_n \propto (k\theta)^{2(n-1)/(n-2)} \tag{19}$$

As to the function φ, some considerations are in order.

First of all φ cannot contain monomial terms in $(\hbar_n \omega^{n-2})/(c^{n-3} k\theta)$, otherwise it would modify the correct depedendence in Eq.(17) of $u_{n,\omega}$ on the term ω^{2n-3} which comes by multiplying the factor ω^{n-2} contained in the photon energy ε_n by the factor ω^{n-1} due to the evaluation of the number of photon states in the interval $d\omega$.

Second, when $n = 3$ we know the behavior of $u_{n,\omega}$ in the limits $\omega \to 0$ (Rayleigh-Jeans's law) and $\omega \to \infty$ (Wien's law), so, when $n = 3$, one must have

$$
\varphi\left(\frac{\hbar\omega}{k\theta}\right) \sim
\begin{cases}
\left(\dfrac{\hbar\omega}{k\theta}\right)^{-1} & \text{for} \quad \omega \to 0 \\[2em]
\exp\left(-\dfrac{\hbar\omega}{k\theta}\right) & \text{for} \quad \omega \to \infty
\end{cases}
\tag{20}
$$

From a strictly mathematical point of view, the function φ which satisfies Eq.(19) is not unique, but, taking into account our requirements, the choice is very restricted and, discarding too involved expressions, one is left only with the function

$$
\varphi\left(\frac{\varepsilon_n}{k\theta}\right) \propto \frac{1}{\exp\left(\dfrac{\varepsilon_n}{k\theta}\right) - 1}
\tag{21}
$$

Once again, we notice that for $n = 3$ Eq.(17) becomes

$$
u_{n,\omega} \propto \frac{\hbar_n \omega^{2n-3}}{c^{2n-3}} \frac{1}{\exp\left(\dfrac{\varepsilon_n}{k\theta}\right) - 1}
\tag{22}
$$

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so that one recovers, apart from a numerical factor $1/\pi^2$, the Planck's formula and hence the Bose-Einstein statistics.

Had we ignored the behavior of the function φ as $\omega \to 0$, the choice made in Eq.(21) might have been changed to $\varphi(\varepsilon_n/(k\theta)) \propto \exp(-\varepsilon_n/(k\theta))$, which is proper for the Boltzmann statistics, or even to $\varphi(\varepsilon_n/(k\theta)) \propto (\exp(\varepsilon_n/(k\theta)) + 1)^{-1}$, which is proper for the Fermi-Dirac statistics.

5 Conclusions

We would like to focus attention on some interesting properties of dimensional analysis we made use of in this paper.

First of all, in treating by the dimensional method a quite general problem of physics, it appeared of capital importance to associate the four fundamental quantities (L, M, T, θ) with as many universal constants, namely $(c, G_n, \hbar_n, k)$. In less general situations, one needs only some of these costants; for example, when restricting to the three-dimensional space, one can select c and G for treating gravity, c and $\hbar$ (or e) for treating electromagnetism or, as in our case, $c, \hbar$ and k for treating the electromagnetic radiation thermodynamics.

On the other hand, such a prescription was alredy contained in the works of Stoney [7] and of Planck [8]. In Planck's words natural units are such because " ... *they are independent of particular bodies or substances, necessarily*

retain their meanings in all times and in every culture even if extra-terrestrial or extra-human." [9]

Secondly, let us comment on the power of the physical dimensions theory to test the soundness of a physical law. To make an example let us consider the Rayleigh-Jeans's formula for black-body radiation. We can obtain it if, contrary to what we just stated, we accept the widespread belief that electromagnetism can be treated by employing as universal constant only c, disregarding $\hbar$ (or e). Acting as before, we would write, in three space dimensions, the new functional relation among the quantities of interest as $f(u_{,\omega}, \omega, \theta, c, k) = 0$. Now we have $r - s = 5 - 4 = 1$ so there is only the following product

$$\Pi = u_{,\omega}^{-1} \omega^{\alpha} \theta^{\beta} c^{\gamma} k^{\delta} \tag{23}$$

The algebraic system is

$$\begin{cases} \alpha + \gamma + 2\delta - 1 = 0 \\ \beta - \delta = 0 \\ \gamma + 2\delta + 1 = 0 \\ \delta - 1 = 0 \end{cases} \tag{24}$$

and its solutions are:

$$\begin{cases} \alpha = 2 \\ \beta = 1 \\ \gamma = -3 \\ \delta = 1 \end{cases} \tag{25}$$

The final result

$$u_{,\omega}(\theta) \propto (k\theta\omega^2)/c^3 \tag{26}$$

reproduces indeed, apart from numerical factors, the Rayleigh-Jeans's formula. As to the Stephan-Boltzmann law one easily realizes that, when removing $\hbar$, Eq.(8) leads to $r - s = 4 - 4 = 0$ and hence no physical law is obtainable, in accordance with the fact that now the energy density $u(\theta)$ blows up to infinity.

Lastly, we wish to remind the reader that a gravitational constant G_n with dimensions $L^n M^{-1} T^{-2}$ is currently used in higher dimensional theories of gravity; so, it seems quite consistent that other universal constants such as e or $\hbar$ be extended to n dimensions! One must take however into account the following. We have seen that the photon energy is $\varepsilon_n = \hbar_n \omega^{n-2}/c^{n-3}$ and hence its momentum is $|\boldsymbol{p}_n| = \varepsilon_n/c = \hbar_n(\omega/c)^{n-2} = \hbar_n |\boldsymbol{k}|^{n-2}$; if now we pass to matter waves and make use of the de Broglie's relations, it happens that only for $n = 3$ the connection is relativistically invariant.

Finally, we would like to add a brief comment about the so-called "black-hole thermodynamics", invented by authors so as Bekenstein [10] and Hawking [11] by generalizing the ordinary thermodynamics.

For instance, a black-body absorbs by definition every electromagnetic radiation impinging on it, and was shown to emit a characteristic (Planck) electromagnetic spectrum. By contrast, a black-hole is defined as a body absorbing every kind of particle and radiation impinging on it, and is predicted to emit a characteristic spectrum for each kind of particle or radiation. Such spectra are expected to be Planck-like, except that in their denominator the minus sign (valid in general for boson emissions) has to be replaced by a plus sign in the case of fermion emissions.

These beautiful results have been reached by a clever, simultaneous recourse to general relativity, quantum mechanics and thermodynamics. They still lack, therefore, a definite theoretical foundation. From this point of view, it may be interesting that they get some support by simple dimensional considerations.

Acknowledgments

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